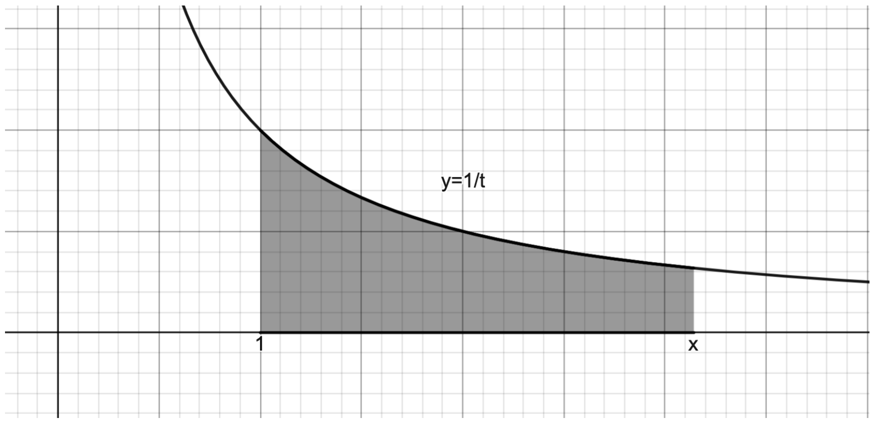
SECTION 7.2 AND 7.3: EXPONENTIAL AND LOGARITHM FUNCTIONS

LOGARITHM FUNCTIONS

QUESTION: Why can't we use the power rule to find $\int \frac{1}{x}\,dx$?

DEFINITION: For $x > 0$, define $\ln(x) = \int_1^x \frac{1}{t}\,dt$.



QUESTIONS:

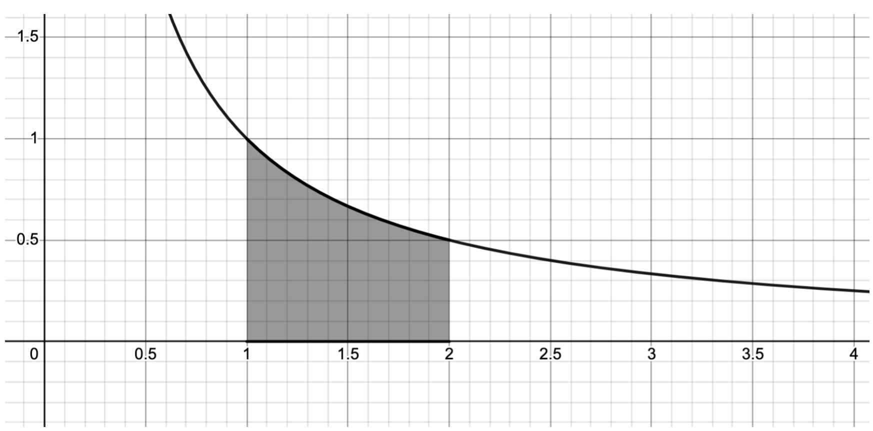
1. Why do we stipulate $x > 0$?

2. What is $\ln(1)$?

3. What is the sign of $\ln(x)$ if $x > 1$?

4. What is the sign of $\ln(x)$ if $0 < x < 1$?

5. Graphically explain why $0.5 < \ln(2) < 1$.



6. What is $D_x[\ln(x)]$? $D_x[\ln(u)]$?

THEOREM: Properties of $f(x) = \ln(x)$.

1. Domain of $f(x) = \ln(x)$ is $(0, \infty)$.

2. $f(x) = \ln(x)$ is increasing and one-to-one.

3. The graph of $y = \ln(x)$ is concave down.

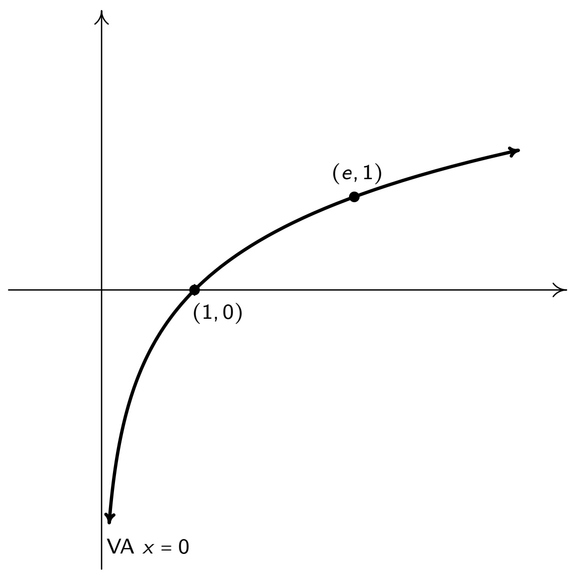
4. $\lim_{x \to 0^+} \ln(x) = -\infty$ and $\lim_{x \to \infty} \ln(x) = \infty$.

5. The range of $f(x) = \ln(x)$ is $(-\infty, \infty)$.

6. $\ln(1) = 0$. Moreover, if $\ln(x) = 0$, then $x = 1$.

7. There is a unique number 'e' with $\ln(e) = 1$. Moreover, $2 < e < 3$.

8. The graph of $y = \ln(x)$ is:



$y = \ln(x)$

9. **PRODUCT TO SUM RULE:** For $M, N > 0$, $\ln(MN) = \ln(M) + \ln(N)$

10. **QUOTIENT TO DIFFERENCE RULE:** For $M, N > 0$, $\ln\left(\frac{M}{N}\right) = \ln(M) - \ln(N)$

11. **POWER TO FACTOR RULE:** For $M > 0$, $\ln\left(M^p\right) = p\ln(M)$

12. **DERIVATIVE FORMULAS:** $D_x[\ln(x)] = \frac{1}{x}$. Moreover, $D_x[\ln|x|] = \frac{1}{x}$ and $D_x[\ln(u)] = \frac{1}{u}u'$.

EXAMPLE 1: For $f(x) = \ln(3x-2)$, find the equation of the tangent line at $x = 1$.

We find $f'(x) = \dfrac{1}{3x-2}\, D_x[3x-2] = \dfrac{3}{3x-2}$. Hence, $f(1) = \ln(3(1)-2) = \ln(1) = 0$ and $f'(1) = \dfrac{3}{3(1)-2} = 3$.

Our tangent line is: $y = f'(1)(x-1) + f(1) = 3(x-1) + 0$ or $y = 3x - 3$.

EXAMPLE 2: Find the indicated derivative.

1. If $y = \dfrac{\ln(x)}{x+1}$, find $\dfrac{dy}{dx}$.

$$\begin{aligned}
\frac{dy}{dx} &= D_x\left[\frac{\ln(x)}{x+1}\right] \\
&= \frac{(x+1)D_x[\ln(x)] - \ln(x)D_x[x+1]}{(x+1)^2} && \text{Quotient Rule} \\
&= \frac{(x+1)\dfrac{1}{x} - \ln(x)(1)}{(x+1)^2} \\
&= \frac{1 + \dfrac{1}{x} - \ln(x)}{(x+1)^2} \cdot \frac{x}{x} && \text{clear complex fractions} \\
\frac{dy}{dx} &= \frac{x + 1 - x\ln(x)}{x(x+1)^2}
\end{aligned}$$

2. Find $f'(x)$ if $f(x) = -\ln(\cos(x))$.

$$\begin{aligned}
f'(x) &= D_x\left[-\ln(\cos(x))\right] \\
&= -D_x\left[\ln(\cos(x))\right] \\
&= -\frac{1}{\cos(x)} D_x[\cos(x)] \quad \text{Chain Rule} \\
&= -\frac{1}{\cos(x)}(-\sin(x)) \\
&= \frac{\sin(x)}{\cos(x)} \\
f'(x) &= \tan(x)
\end{aligned}$$

3. Find $D_x\left[x\tan^{-1}(x) - \frac{1}{2}\ln\left(x^2+1\right)\right]$

$$\begin{aligned}
D_x\left[x\tan^{-1}(x) - \frac{1}{2}\ln\left(x^2+1\right)\right] &= D_x\left[x\tan^{-1}(x)\right] - \frac{1}{2}D_x\left[\ln\left(x^2+1\right)\right] \\
&= D_x[x]\tan^{-1}(x) + x\,D_x\left[\tan^{-1}(x)\right] - \frac{1}{2}\left(\frac{1}{x^2+1}\right)D_x\left[x^2+1\right] \\
&= (1)\tan^{-1}(x) + x\left(\frac{1}{1+x^2}\right) - \frac{1}{2}\left(\frac{1}{x^2+1}\right)(2x) \\
&= \tan^{-1}(x) + \frac{x}{x^2+1} - \frac{x}{x^2+1} \\
D_x\left[x\tan^{-1}(x) - \frac{1}{2}\ln\left(x^2+1\right)\right] &= \tan^{-1}(x)
\end{aligned}$$

EXAMPLE 3: (VIDEO) Find the indicated derivative.

1. For $y = \ln(5-x)$, find y'.

Ans: $y' = \dfrac{1}{x-5}$

2. Find and simplify: $D_\theta\left[\ln(\sec(\theta)+\tan(\theta))\right]$.

Ans: $D_\theta\left[\ln(\sec(\theta)+\tan(\theta))\right] = \sec(\theta)$.

EXAMPLE 4: Find $\dfrac{dy}{dx}$ for $y = \ln\left(\dfrac{x^3\sqrt{3x-1}}{x-5}\right)$. Assume $x > 5$.

Since we are told to assume $x > 5$, we know all the quantities x, x^3, $(3x-1)$ and $(x-5)$ are positive. Hence, we may use properties of logarithms before we differentiate to greatly simplify our workflow.

$$\begin{aligned}
y &= \ln\left(\frac{x^3\sqrt{3x-1}}{x-5}\right) \\
&= \ln(x^3\sqrt{3x-1}) - \ln(x-5) && \text{Quotient to Difference Rule} \\
&= \ln(x^3) + \ln\sqrt{3x-1} - \ln(x-5) && \text{Product to Sum Rule} \\
&= 3\ln(x) + \ln(3x-1)^{\frac{1}{2}} - \ln(x-5) && \text{Power to Factor Rule} \\
y &= 3\ln(x) + \tfrac{1}{2}\ln(3x-1) - \ln(x-5) && \text{Power to Factor Rule}
\end{aligned}$$

Taking derivatives, we get:

$$\begin{aligned}
\frac{dy}{dx} &= D_x[3\ln(x) + \tfrac{1}{2}\ln(3x-1) - \ln(x-5)] \\
&= 3\,D_x[\ln(x)] + \tfrac{1}{2}\,D_x\,[\ln(3x-1)] - D_x[\ln(x-5)] \\
&= 3\left(\frac{1}{x}\right) + \frac{1}{2}\left(\frac{1}{3x-1}\right) D_x[3x-1] - \left(\frac{1}{x-5}\right) D_x[x-5] \quad \text{Chain Rule} \\
\frac{dy}{dx} &= \frac{3}{x} + \frac{3}{6x-2} - \frac{1}{x-5}
\end{aligned}$$

EXAMPLE 5: Let $y = x^{\sin(x)}$, for $x > 0$. Find $\dfrac{dy}{dx}$.

The function $y = x^{\sin(x)}$ doesn't fit into any of our formulas for differentiation (do you see why?)

To find $\dfrac{dy}{dx}$, we employ a special kind of implicit differentiation called **logarithmic differentiation**.

We take logs and make use of log properties to rewrite the function implicitly before taking the derivative.

$$\begin{aligned}
y &= x^{\sin(x)} \\
\ln(y) &= \ln\left(x^{\sin(x)}\right) \\
\ln(y) &= \sin(x)\,\ln(x) && \text{Power to Factor Rule} \\
D_x[\ln(y)] &= D_x[\sin(x)\,\ln(x)] \\
\frac{1}{y}\,D_x[y] &= D_x[\sin(x)]\,\ln(x) + \sin(x)\,D_x[\ln(x)] && \text{Chain and Product Rules} \\
\frac{1}{y}\frac{dy}{dx} &= \cos(x)\,\ln(x) + \sin(x)\,\frac{1}{x} \\
\frac{dy}{dx} &= y\left(\cos(x)\,\ln(x) + \frac{\sin(x)}{x}\right) \\
\frac{dy}{dx} &= x^{\sin(x)}\left(\cos(x)\,\ln(x) + \frac{\sin(x)}{x}\right)
\end{aligned}$$

STEPS FOR LOGARITHMIC DIFFERENTIATION

1. Write the function as $y = f(x)$.

2. Take the natural log of both sides of the equation: $\ln(y) = \ln(f(x))$.

3. Use properties of logarithms to rewrite $\ln(f(x))$.

4. Differentiate both sides of the equation $\ln(y) = \ln(f(x))$ with respect to x.

   Remember the chain rule: $D_x[\ln(y)] = \dfrac{1}{y}y'$

5. Solve for y' by multiplying: $y' = y\,D_x[\ln(f(x))]$.

EXAMPLE 6: (VIDEO) Let $y = x^{3x}$, for $x > 0$. Find $\dfrac{dy}{dx}$ using logarithmic differentiation.

Ans: $\dfrac{dy}{dx} = x^{3x}(3\ln(x) + 3)$

NOTE: Using logarithmic differentiation on $y = x^k$ where k is any **real** number results in the power rule:

For all **real** numbers k, $D_x\left[x^k\right] = k\,x^{k-1}$

Hence $D_x\left[x^{\sqrt{2}}\right] = \sqrt{2}\,x^{\sqrt{2}-1}$, $D_x[x^\pi] = \pi x^{\pi-1}$, etc.

LOGARITHM FUNCTIONS WITH BASES OTHER THAN 'e':

We have by definition that $\ln(x) = y$ means $e^y = x$.

What if we wanted to change the base of the log from 'e' to some other base 'b' where $b > 0$, $b \neq 1$?

If $y = \log_b(x)$, then $b^y = x$. Taking natural logs, we get $\ln(b^y) = \ln(x)$ so that $y\ln(b) = \ln(x)$ or $y = \dfrac{\ln(x)}{\ln(b)}$.

Hence we **define** $\log_b(x) = \dfrac{\ln(x)}{\ln(b)}$. (You may have learned this in pre-calculus as the 'change of base formula.')

From the change of base, Hence,

$$D_x[\log_b(x)] = D_x\left[\frac{\ln(x)}{\ln(b)}\right] = \frac{1}{\ln(b)}D_x[\ln(x)] = \frac{1}{\ln(b)}\left(\frac{1}{x}\right) = \frac{1}{x\ln(b)}$$

So, for example, $D_x[\log_2(x)] = \dfrac{1}{x\ln(2)}$, $D_x[\log(x)] = D_x[\log_{10}(x)] = \dfrac{1}{x\ln(10)}$, etc.

EXAMPLE 7: (VIDEO) Find the equation of the tangent line to the curve $y = \log_2(3x+1)$ at $x = 1$.

Ans: $y = \dfrac{1}{4\ln(2)}x + 2 - \dfrac{1}{4\ln(2)}$

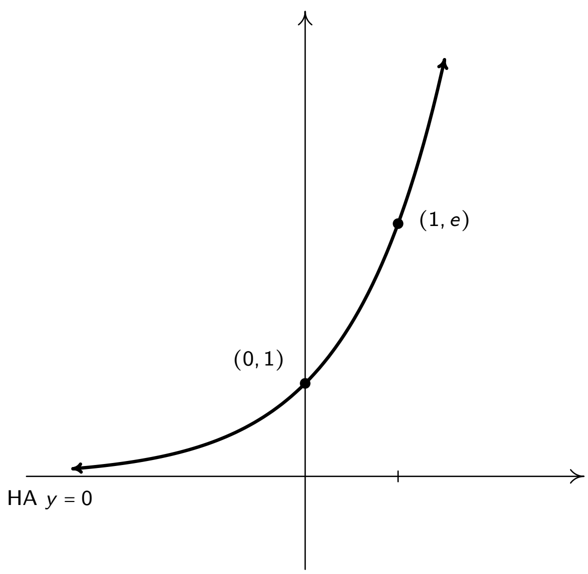
EXPONENTIAL FUNCTIONS

Since $f(x) = \ln(x)$ is one-to-one, there is an inverse function, $g(x)$ so that $f(g(x)) = \ln(g(x)) = x$.

Since $\ln(e^x) = x\ln(e) = x$, we have $\ln(g(x)) = \ln(e^x)$. Since $f(x) = \ln(x)$ is one-to-one, we have: $g(x) = e^x$.

THEOREM: Properties of $g(x) = e^x$ (as inherited from $f(x) = \ln(x)$):

1. **INVERSE PROPERTIES:** $\ln(e^x) = x$ for all real numbers x and $e^{\ln(x)} = x$ for all $x > 0$.
2. Domain of $g(x) = e^x$ is $(-\infty, \infty)$.
3. $g(x) = e^x$ is increasing and one-to-one.
4. The graph of $y = e^x$ is concave up.
5. $\lim_{x\to-\infty} e^x = 0$ and $\lim_{x\to\infty} e^x = \infty$.
6. The range of $f(x) = e^x$ is $(0, \infty)$. Said differently, $e^x > 0$ for all real numbers, x.
7. $e^0 = 1$. Moreover, if $e^x = 1$, then $x = 0$.
8. The graph of $y = e^x$ is:



$y = e^x$

9. **PRODUCT TO SUM RULE:** For any two real numbers m and n, $e^m\, e^n = e^{m+n}$.
10. **QUOTIENT TO DIFFERENCE RULE:** For any two real numbers m and n, $\dfrac{e^m}{e^n} = e^{m-n}$.
11. **POWER TO FACTOR RULE:** For any two real numbers m and p, $(e^m)^p = e^{mp}$.

DERIVATIVE FORMULA FOR EXPONENTIALS: $D_x[e^x] = e^x$. More generally, $D_x[e^u] = e^u\, u'$.

PROOF: For $y = e^x$, $\ln(y) = x$ so $\dfrac{1}{y}\, y' = 1$ or $y' = y$. Hence, $D_x[e^x] = e^x$.

EXAMPLE 8: Find the indicated derivatives.

1. $D_x\left[x\,e^x - e^x\right]$

$$\begin{aligned}
D_x\left[x\,e^x - e^x\right] &= D_x\left[x\,e^x\right] - D_x\left[e^x\right] && \text{Sum and Difference Rule}\\
&= D_x[x]e^x + x\,D_x\left[e^x\right] - e^x && \text{Product Rule}\\
&= (1)e^x + x\,e^x - e^x\\
&= e^x + x\,e^x - e^x\\
D_x\left[x\,e^x - e^x\right] &= x\,e^x
\end{aligned}$$

2. For $A(t) = 50e^{-t}$, find $A'(t)$.

$$\begin{aligned}
A'(t) &= D_t[A(t)]\\
&= D_t[50e^{-t}]\\
&= 50D_t[e^{-t}]\\
&= 50e^{-t}D_t[-t] \quad \text{Chain Rule}\\
&= 50e^{-t}(-1)\\
A'(t) &= -50e^{-t}
\end{aligned}$$

EXAMPLE 9: If $y = e^{2x}\sin(3x)$, find $\dfrac{dy}{dx}$.

$$\begin{aligned}
\frac{dy}{dx} &= D_x\left[e^{2x}\sin(3x)\right]\\
&= D_x\left[e^{2x}\right]\sin(3x) + e^{2x}D_x\left[\sin(3x)\right] && \text{Product Rule}\\
&= e^{2x}D_x[2x]\sin(3x) + e^{2x}\cos(3x)D_x[3x] && \text{Chain Rule}\\
&= e^{2x}(2)\sin(3x) + e^{2x}\cos(3x)(3)\\
\frac{dy}{dx} &= 2e^{2x}\sin(3x) + 3e^{2x}\cos(3x)
\end{aligned}$$

EXAMPLE 10: Find the equation to the tangent line to the graph of $f(x) = e^x$ at $x = 0$.

We have $f'(x) = e^x$ so $f(0) = f'(0) = e^0 = 1$.

Hence, the tangent line is: $y = f'(0)(x - 0) = f(0) = (1)(x) + 1$ or $y = x + 1$.

EXAMPLE 11: (VIDEO) Find the indicated derivative.

1. If $T(t) = 50 + 10e^{-0.25t}$, find $T'(t)$.

Ans: $T'(t) = -2.5e^{-0.25t}$

2. If $P(t) = \dfrac{5000}{1 + 10e^{-0.1t}}$, find $\dfrac{dP}{dt}$.

Ans: $\dfrac{dP}{dt} = \dfrac{5000e^{-0.1t}}{(1 + 10e^{-0.1t})^2}$

3. For $f(x) = \dfrac{e^x + e^{-x}}{2}$, find $f''(x)$. Are you surprised?

Ans: $f''(x) = \dfrac{e^x + e^{-x}}{2} = f(x)$

EXAMPLE 12: Prove that if k is a constant, then $D_x\left[e^{kx}\right] = k\,e^{kx}$.

Using the chain rule, we get: $D_x\left[e^{kx}\right] = e^{kx}\,D_x\left[kx\right] = e^{kx}\,k = k\,e^{kx}\,\checkmark$.

EXAMPLE 13: The number of yeast organisms in suspension, $N(t)$, t minutes after being introduced is:

$$N(t) = 5000e^{0.0077t}$$

1. Find and interpret $N(0)$ and $N'(0)$.

   $N(0) = 5000e^{0.0077(0)} = 5000$. This means that initially, there are 5000 yeast introduced to the suspension.

   We find $N'(t) = 5000e^{0.0077t}D_t[0.0077t] = 5000e^{0.0077t}(0.0077) = 38.5e^{0.0077t}$.

   Hence, $N'(0) = 38.5$. Initially, the population of yeast is increasing at a rate of 38.5 organisms per minute.

2. Find and interpret $\dfrac{N'(t)}{N(t)}$.

   $\dfrac{N'(t)}{N(t)} = \dfrac{38.5e^{0.0077t}}{5000e^{0.0077t}} = 0.0077$.

   Here, we are dividing the **rate** of population growth at time t by the **population** at t.

   This gives us a **relative** rate of change: the population is increasing at a **constant relative rate** of 0.77 %.

EXPONENTIAL FUNCTIONS WITH BASES OTHER THAN 'e':

Using the properties of exponents and logs, we get: $e^{x\ln(b)} = e^{\ln(b^x)} = b^x$. Hence,

$$D_x\left[b^x\right] = D_x\left[e^{x\ln(b)}\right] = e^{x\ln(b)} D_x[x\ln(b)] = e^{x\ln(b)}\ln(b) = b^x\ln(b)$$

So, for example, $D_x\left[3^x\right] = 3^x\ln(3)$, $D_t\left[(1.025)^t\right] = (1.025)^t\ln(1.025)$, etc.

EXAMPLE 14: (VIDEO) If $f(x) = x^3\,3^x$, find $f'(x)$.

Ans: $f'(x) = 3x^2\,3^x + x^3\,3^x\ln(3)$

EXAMPLE 15: (VIDEO) The amount of money in a savings account after t years is modeled by

$$A(t) = 500(1.002)^{12t}.$$

1. Find and interpret $A(0)$ and $A'(0)$.

   Ans: $A(0) = 500(1.002)^{12(0)} = 500$. This means the initial investment is \$500.

   $A'(0) = 6000\ln(1.002) \approx 12$. Initially, the savings account is growing at \$12 per year.

2. Find and interpret $\dfrac{A'(0)}{A(0)}$.

   Ans: $\dfrac{A'(0)}{A(0)} \approx \dfrac{12}{500} = 0.024$ so the initial relative growth rate is 2.4%.

3. Find and interpret $\dfrac{A'(t)}{A(t)}$.

   Ans: $\dfrac{A'(t)}{A(t)} = \dfrac{6000(1.002)^{12t}\ln(1.002)}{500(1.002)^{12t}} = 12\ln(1.002) \approx 0.024$

   Hence, the relative growth rate is a constant 2.4% per year.

INTEGRALS INVOLVING EXPONENTIAL AND LOGARITHM FUNCTIONS

RECALL: If $b > 0$, $b \neq 1$:

- $D_x\left[e^x\right] = e^x$
- $D_x\left[b^x\right] = b^x \ln(b)$
- $D_x\left[\ln(x)\right] = \dfrac{1}{x}$

RULE OF THUMB: Let u = what's in parentheses.

EXAMPLE 16: Find the following integrals. Check your answer using differentiation.

1. $\displaystyle\int \cos\left(e^{3x}\right) e^{3x}\, dx$

   Let $u = e^{3x}$. Then $du = 3e^{3x}\, dx$ so:

$$\begin{aligned}
\int \cos\left(e^{3x}\right) e^{3x}\, dx &= \int \cos\left(e^{3x}\right)\left(\frac{3}{3}\right) e^{3x}\, dx \\
&= \frac{1}{3}\int \cos\left(e^{3x}\right) 3e^{3x}\, dx \\
&= \frac{1}{3}\int \cos(u)\, du \\
&= \frac{1}{3}\sin(u) + C \\
\int \cos\left(e^{3x}\right) e^{3x}\, dx &= \frac{1}{3}\sin\left(e^{3x}\right) + C
\end{aligned}$$

   To check, we find $D_x\left[\frac{1}{3}\sin\left(e^{3x}\right)\right] = \ldots = \cos\left(e^{3x}\right) e^{3x}$ ✓

2. $\displaystyle\int \frac{\sqrt{\ln(x)+9}}{x}\, dx$

   Rewriting, we have: $\displaystyle\int \frac{\sqrt{\ln(x)+9}}{x}\, dx = \int \frac{(\ln(x)+9)^{\frac{1}{2}}}{x}\, dx$. We let $u = \ln(x) + 9$ so $du = \dfrac{1}{x}\, dx$. Hence:

$$\begin{aligned}
\int \frac{(\ln(x)+9)^{\frac{1}{2}}}{x}\, dx &= \int (\ln(x)+9)^{\frac{1}{2}}\,\frac{1}{x}\, dx \\
&= \int u^{\frac{1}{2}}\, du \\
&= \frac{2}{3}u^{\frac{3}{2}} + C \\
\int \frac{\sqrt{\ln(x)+9}}{x}\, dx &= \frac{2}{3}(\ln(x)+9)^{\frac{3}{2}} + C
\end{aligned}$$

   To check, we find: $D_x\left[\frac{2}{3}(\ln(x)+9)^{\frac{3}{2}} + C\right] = \ldots = \dfrac{\sqrt{\ln(x)+9}}{x}$ ✓

EXAMPLE 17: (VIDEO) Find the following integrals. Check your answer using differentiation.

1. $\displaystyle\int \frac{2^x}{\sqrt{2^x+1}}\,dx$

   Ans: $\displaystyle\int \frac{2^x}{\sqrt{2^x+1}}\,dx = \frac{2}{\ln(2)}\left(2^x+1\right)^{\frac{1}{2}} + C$

2. $\displaystyle\int \frac{\cos\left(e^{-t}\right)}{e^t}\,dt$

   Ans: $\displaystyle\int \frac{\cos\left(e^{-t}\right)}{e^t}\,dt = -\sin\left(e^{-t}\right) + C$

3. $\displaystyle\int \frac{\sec(\ln(x))\tan(\ln(x))}{x}\,dx$

   Ans: $\displaystyle\int \frac{\sec(\ln(x))\tan(\ln(x))}{x}\,dx = \sec(\ln(x)) + C$

4. $\displaystyle\int \ln(\sin(\theta))\cot(\theta)\,d\theta$

   HINT: Let $u = \ln(\sin(\theta))$ ...

   Ans: $\displaystyle\int \ln(\sin(\theta))\cot(\theta)\,d\theta = \frac{1}{2}\left[\ln(\sin(\theta))\right]^2 + C$

RECALL: For $b > 0$, $b \neq 1$:

- $\int e^u \, du = e^u + C$
- $\int b^u \, du = \dfrac{1}{\ln(b)} b^u + C$
- $\int \dfrac{1}{u} \, du = \ln|u| + C$

MORE RULES OF THUMB: Let u = what's in the exponent or let u = what's in the denominator.

EXAMPLE 18: Find the following integrals. Check your answer using differentiation.

1. $\displaystyle\int 3^{\tan(x)} \sec^2(x) \, dx$

   We let $u = \tan(x)$ so that $du = \sec^2(x)\, dx$. Then:

   $$\int 3^{\tan(x)} \sec^2(x) \, dx = \int 3^u \, du = \frac{1}{\ln(3)} 3^u + C = \frac{1}{\ln(3)} 3^{\tan(x)} + C$$

   To check, we find $D_x\left[\frac{1}{\ln(3)} 3^{\tan(x)} + C\right] = \ldots = 3^{\tan(x)} \sec^2(x)$ ✓

2. $\displaystyle\int \tan(\theta) \, d\theta$

   We rewrite $\displaystyle\int \tan(\theta)\, d\theta = \int \frac{\sin(\theta)}{\cos(\theta)}\, d\theta$. We let $u = \cos(\theta)$ so $du = -\sin(\theta)\, d\theta$. Then:

   $$\int \tan(\theta)\, d\theta = \int \frac{\sin(\theta)}{\cos(\theta)}\, d\theta = -\int \frac{1}{\cos(\theta)} \left(-\sin(\theta)\right)\, d\theta = -\int \frac{1}{u}\, du = -\ln|u| + C = -\ln|\cos(\theta)| + C$$

   To check, we find $D_\theta\left[-\ln|\cos(\theta)| + C\right] = \ldots = \tan(\theta)$ ✓

3. $\displaystyle\int \frac{1}{x \ln(x)}\, dx$

   After some trial and error, we choose $u = \ln(x)$ so $du = \dfrac{1}{x}\, dx$. Hence:

   $$\int \frac{1}{x\ln(x)}\, dx = \int \frac{1}{\ln(x)} \frac{1}{x}\, dx = \int \frac{1}{u}\, du = \ln|u| + C = \ln|\ln(x)| + C$$

   We check: $D_x\left[\ln|\ln(x)| + C\right] = \ldots = \dfrac{1}{x\ln(x)}$ ✓

EXAMPLE 19: (VIDEO) Find the following integrals. Check your answer using differentiation.

1. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

Ans: $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2e^{\sqrt{x}} + C$

2. $\displaystyle\int \frac{1}{2x+1}\,dx$

Ans: $\displaystyle\int \frac{1}{2x+1}\,dx = \frac{1}{2}\ln|2x+1| + C$

3. $\displaystyle\int \frac{e^t - e^{-t}}{e^t + e^{-t}}\,dt$

Ans: $\displaystyle\int \frac{e^t - e^{-t}}{e^t + e^{-t}}\,dt = \ln\left(e^t + e^{-t}\right) + C$

4. $\displaystyle\int \cot(\theta)\,d\theta$

Ans: $\displaystyle\int \cot(\theta)\,d\theta = \ln|\sin(\theta)| + C$

EXAMPLE 20: Use long division to help you find: $\displaystyle\int \frac{3x-2}{2x+1}\,dx$.

Using long division, we get: $\dfrac{3x-2}{2x+1} = \dfrac{3}{2} - \dfrac{7}{2(2x+1)}$. Hence:

$$\begin{aligned}
\int \frac{3x-2}{2x+1}\,dx &= \int \left[\frac{3}{2} - \frac{7}{2(2x+1)}\right] dx \\
&= \int \frac{3}{2}\,dx - \frac{7}{2}\int \frac{1}{2x+1}\,dx \\
&= \frac{3}{2}x - \left(\frac{7}{2}\right)\left(\frac{1}{2}\right)\int \frac{1}{u}\,du \qquad \text{Let } u = 2x+1 \text{ so } du = 2\,dx \ldots \\
\int \frac{3x-2}{2x+1}\,dx &= \frac{3}{2}x - \frac{7}{4}\ln|2x+1| + C
\end{aligned}$$

EXAMPLE 21: (VIDEO) Find: $\displaystyle\int \frac{x^2+2}{x-1}\,dx$.

Ans: $\displaystyle\int \frac{x^2+2}{x-1}\,dx = \int \left(x + 1 + \frac{3}{x-1}\right) dx = \frac{1}{2}x^2 + x + 3\ln|x-1| + C$

EXAMPLE 22: The rate of change of the value of a car (in dollars per year) t years after it is purchased is:

$$V'(t) = -2000e^{-0.2t}, \quad t \geq 0.$$

1. Find and interpret $\int_0^5 V'(t)\,dt$.

$$\begin{aligned}
\int_0^5 V'(t)\,dt &= \int_0^5 \left(-2000e^{-0.2t}\right) dt \\
&= -2000 \int_0^5 e^{-0.2t}\,dt \\
&= -\frac{2000}{-0.2} e^{-0.2t}\Big|_{t=0}^{t=5} = 10000e^{-0.2t}\Big|_{t=0}^{t=5} \\
&= 10000e^{-0.2(5)} - 10000e^{-0.2(0)} \\
&= 10000e^{-1} - 10000 \approx -6321.21
\end{aligned}$$

   Over the first five years of ownership, the car has lost approximately \$6,321.21 of its value.

2. If the initial value of the car was \$10,000, find a formula for the value of the car, $V(t)$.

$$\begin{aligned}
V(t) &= V(0) + \int_0^t V'(u)\,du \\
&= 10000 + \int_0^t \left(-2000e^{-0.2u}\right) du \\
&= 10000 - \frac{2000}{-0.2} e^{-0.2u}\Big|_{u=0}^{u=t} \\
&= 10000 + 10000e^{-0.2u}\Big|_{u=0}^{u=t} \\
&= 10000 + 10000e^{-0.2t} - 10000e^{0.2(0)} \\
&= 10000 + 10000e^{-0.2t} - 10000 \\
V(t) &= 10000e^{-0.2t}
\end{aligned}$$

EXAMPLE 23: (VIDEO) The growth rate of bacteria (in organisms per minute) after t minutes is

$$P'(t) = 35e^{0.007t}, \quad t \geq 0.$$

1. Find and interpret $\int_0^{60} P'(t)\,dt$.

   Ans: $\int_0^{60} P'(t)\,dt = 5000e^{0.42} - 5000 \approx 2610.$

   After 1 hour (60 minutes) the population of bacteria has increased by 2610.

2. If the initial number of bacteria is 5000, find a formula for the population of bacteria, $P(t)$.

   Ans: $P(t) = P(0) + \int_0^t 35e^{0.007u}\,du = \ldots = 5000e^{0.007t}$

EXAMPLE 24: Find: $\displaystyle\int \frac{1}{\sqrt{e^{2x}-1}}\,dx$

HINT: Multiply the integrand by $\dfrac{e^x}{e^x}$

We rewrite: $\displaystyle\int \frac{1}{\sqrt{e^{2x}-1}}\,dx = \int \frac{e^x}{e^x\sqrt{e^{2x}-1}}\,dx = \int \frac{e^x}{e^x\sqrt{(e^x)^2-1}}\,dx$. We let $u = e^x$ so $du = e^x\,dx$.

Hence: $\displaystyle\int \frac{e^x}{e^x\sqrt{(e^x)^2-1}}\,dx = \int \frac{1}{e^x\sqrt{(e^x)^2-1}}\,e^x\,dx = \int \frac{1}{u\sqrt{u^2-1}}\,du = \sec^{-1}|u| + C = \sec^{-1}|e^x| + C.$

Since $e^x > 0$, $|e^x| = e^x$ so $\displaystyle\int \frac{1}{\sqrt{e^{2x}-1}}\,dx = \sec^{-1}(e^x) + C.$

To check, we find: $D_x\left[\sec^{-1}(e^x) + C\right] = \ldots = \dfrac{1}{\sqrt{e^{2x}-1}}$ ✓

EXAMPLE 25: (VIDEO) Find: $\displaystyle\int \frac{1}{e^t + e^{-t}}\,dt$

HINT: Multiply the integrand by $\dfrac{e^t}{e^t}$.

Ans: $\displaystyle\int \frac{1}{e^t + e^{-t}}\,dt = \tan^{-1}\left(e^t\right) + C$

CHALLENGE! Find: $\displaystyle\int \frac{1}{1+\sqrt{x}}\,dx$.

Ans: $\displaystyle\int \frac{1}{1+\sqrt{x}}\,dx = 2\sqrt{x} - 2\ln\left(1+\sqrt{x}\right) + C$

HOMEWORK:

Section 7.2:

- Algebra Review: 7 - 13 odd
- Calculus Problems: 17 - 37 odd (derivatives); 41 - 59 odd (integrals), 63 - 69 odd (logarithmic differentiation), 77 - 95 odd (mix), 105

Section 7.3:

- Algebra Review: 11 - 21 odd
- Calculus Problems: 23 - 33 odd (derivatives), 39 - 43 odd (integrals), 45 - 71 odd, 81 - 103 odd (mix)